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LETTERS TO THE EDITOR

CALCULATION OF NORMALIZATION INTEGRAL OF BEAM FUNCTIONS

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1. INTRODUCTION

The method of eigenfunction expansion or modal analysis is an efficient tool for analyzing the vibration problem of beam and plate structures. The free vibration analysis determines both of modal frequencies and their corresponding mode functions for the associated boundary conditions. Such functions have the relation of orthogonality in the sense of the energy inner product if the structural system has suitable properties [1]. When analyzing the forced vibration problem using modal analysis, the normalization integral of mode functions should be evaluated. Although it is possible to perform the integration analytically or numerically, the procedures involved in either case are quite complicated and tedious.

In references [2, 3], a useful formula for the normalization integral of the beam functions of a uniform classical (Euler–Bernoulli) beam is shown; the integral is expressed in terms of the boundary values of the beam function and its derivatives of higher order. There is surely another formula, which often used in dynamic analysis of the beam-like structures [4–6], for the normalization integral as follows:

$$\int \phi^2 \,\mathrm{d}\xi = \frac{1}{4\lambda^4}(3\phi\phi''' + \phi'\phi'') + \frac{1}{2}(A^2 + B^2 + C^2 - D^2)\xi,$$

where a prime denotes the differentiation with respect to ξ, and ϕ is the solution of $\mathrm{d}^4\phi/\mathrm{d}\xi^4 = \lambda^4\phi$ with the associated boundary conditions and is assumed to have the following form:

$$\phi = A\cos\lambda\xi + B\sin\lambda\xi + C\cosh\lambda\xi + D\sinh\lambda\xi.$$

Neither of these normalization formulae is directly applicable even to the beam subjected to a time-invariant constant axial force. Furthermore, an equivalent formulas has not been known yet for the first order (Timoshenko) beam theory which includes the effects of both rotatory inertia and shear deformation; orthogonality of the mode functions of Timoshenko beam was confirmed [7].

The present paper provides a straightforward method, which may be regarded as an extension method of that in references [2, 3], for evaluating the normalization integral of the characteristic beam functions of a uniform beam. It is assumed here that the beam obeys the beam theories up to the first order, namely, Timoshenko beam theory. Two applications for the present method are shown.

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2. PREPARATION

Here, we show fundamental relations which will be used for evaluating the scalar product of two mode functions of a uniform beam. We assume that the fundamental differential equation for the beam in a steady state of free vibration has the following expression:

$$\frac{\mathrm{d}^4\phi}{\mathrm{d}\xi^4} + f(\omega)\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + g(\omega)\phi = 0, \tag{1}$$

where ϕ is the spatial function of the beam deflection to be determined in practical problems, the coefficients f and g are certain functions dependent on only the parameter ω, and ξ denotes a spatial co-ordinate along the beam span. The parameter ω corresponds to the circular frequency of the system under investigation. It is also to be noted that equation (1) can generally represent the free vibration of beams obeying beam theories up to the first order, namely, Euler–Bernoulli beam, Rayleigh beam and Timoshenko beam theories. Since the solution ϕ of equation (1) will be a function continuously dependent on both of the spatial co-ordinate ξ and the parameter ω, the first derivative of ϕ with respect to ω is expressed as follows:

$$\phi_\omega = \frac{\xi}{2g(4g - f^2)}\{(2gg_\omega + (gf_\omega - fg_\omega)f)\phi' + (2gf_\omega - fg_\omega)\phi'''\}, \tag{2}$$

where the subscript ω denotes the differentiation with respect to the parameter ω. It can be easily confirmed that the parametric and spatial differentiations are commutative, thus resulting in the following relations:

$$\phi'_\omega = \frac{\xi}{2(4g - f^2)}\{(fg_\omega - 2gf_\omega)\phi + (2g_\omega - ff_\omega)\phi''\} + \frac{1}{2g(4g - f^2)}\{(2gg_\omega + (gf_\omega - fg_\omega)f)\phi' + (2gf_\omega - fg_\omega)\phi'''\}, \tag{3}$$

$$\phi''_\omega = \frac{\xi}{2(4g - f^2)}\{(fg_\omega - 2gf_\omega)\phi' + (2g_\omega - ff_\omega)\phi'''\} + \frac{1}{4g - f^2}\{(fg_\omega - 2gf_\omega)\phi + (2g_\omega - ff_\omega)\phi''\} \tag{4}$$

and

$$\phi'''_\omega = \frac{\xi}{2(4g - f^2)}\{g(ff_\omega - 2g_\omega)\phi + ((f^2 - 2g)f_\omega - fg_\omega)\phi''\} + \frac{3}{2(4g - f^2)}\{(fg_\omega - 2gf_\omega)\phi' + (2g_\omega - ff_\omega)\phi'''\}. \tag{5}$$

Although all the above expressions become indefinite when $g = 0$ or $4g - f^2 = 0$, we can rule out such conditions because they seldom occur in practical problems. The above relations will be used to evaluate an indeterminate expression with the help of L'Hospital's rule in the following section.

3. EXAMPLES

As a first application, we consider the transverse vibration problem of a uniform Euler–Bernoulli beam of finite length l. The beam is supposed to rest on a uniform elastic (Winkler) foundation of the stiffness K and to be subjected to a time-invariant constant axial force P. When considering the beam in a state of free vibration, we have the equation of motion as follows:

$$EI\frac{\partial^4 w}{\partial x^4} - P\frac{\partial^2 w}{\partial x^2} + Kw + \rho A\frac{\partial^2 w}{\partial t^2} = 0, \tag{6}$$

where E is the Young's modulus, I is the second moment of the cross-section, ρ is the mass density, A is the cross-sectional area, w is the transverse deflection, and t is time. The axial force P is supposed to be conservative, to be below the lowest buckling load and to take a positive sign for tensile here. The associated boundary conditions are, therefore, supposed to be at $x = 0$,

$$EI\frac{\partial^2 w}{\partial x^2} - K_{\theta 0}\frac{\partial w}{\partial x} = 0, \qquad EI\frac{\partial^3 w}{\partial x^3} - P\frac{\partial w}{\partial x} + K_{T0}w = 0, \tag{7}$$

and at $x = l$,

$$EI\frac{\partial^2 w}{\partial x^2} + K_{\theta 1}\frac{\partial w}{\partial x} = 0, \qquad EI\frac{\partial^3 w}{\partial x^3} - P\frac{\partial w}{\partial x} - K_{T1}w = 0. \tag{8}$$

where $K_{\theta 0}$, K_{T0}, $K_{\theta 1}$ and K_{T1} are the rotational spring and translational spring constants at the left and right ends of the beam, respectively. Here, we introduce the dimensionless quantities such as

$$\xi = x/l, \quad p = Pl^2/EI, \quad k = Kl^4/EI, \quad \tau = (EI/\rho A l^4)^{1/2}t,$$

$$k_{\theta 0} = K_{\theta 0}l/EI, \quad k_{T0} = K_{T0}l^3/EI, \quad k_{\theta 1} = K_{\theta 1}l/EI, \quad k_{T1} = K_{T1}l^3/EI. \tag{9}$$

Then, the governing differential equation (6) can be rewritten in the form of

$$\frac{\partial^4 w}{\partial \xi^4} - p\frac{\partial^2 w}{\partial \xi^2} + kw + \frac{\partial^2 w}{\partial \tau^2} = 0. \tag{10}$$

The associated boundary conditions become

$$w'' - k_{\theta 0}w' = 0, \quad w''' - pw' + k_{T0}w = 0 \tag{11}$$

and

$$w'' + k_{\theta 1}w' = 0, \quad w''' - pw' - k_{T1}w = 0. \tag{12}$$

Substituting $w = \phi(\xi)e^{i\omega\tau}$ into equation (10) yields

$$\phi'''' - p\phi'' + (k - \omega^2)\phi = 0, \tag{13}$$

where $\phi(\xi)$, i and ω denotes the spatial function, the imaginary unit and the dimensionless circular frequency respectively. Now, equation (13) is equivalent to equation (1) with

$f = -p$ and $g = k - \omega^2$. It should be noticed that the solution ϕ of equation (13) will change its expression in accordance witht he sign of $p^2 + 4\omega^2 - 4k$. When putting $\omega = \omega_i$ in equation (13), we can find the solution ϕ_i corresponding to ω_i. Then, carrying out the ordinary procedure for orthogonality relation, we obtain the following expression:

$$\int_0^1 \phi_i \phi_j \, \mathrm{d}\xi = \frac{[\phi_i(\phi_j''' - p\phi_j') - \phi_i'\phi_j'' + \phi_i''\phi_j' - (\phi_i''' - p\phi_i')\phi_j]_0^1}{\omega_j^2 - \omega_i^2} \tag{14}$$

When the functions ϕ_i and ϕ_j satisfy the associated boundary conditions, the right-hand side of equation (14) vanishes. Consequently, orthogonality is established under the condition that ω_i and ω_j are distinct. For $\omega_j = \omega_i$, however, the right-hand side of equation (14) becomes indefinite. Then, taking the limit of equation (14) as $\omega_j \to \omega_i$, with the aid of L'Hospital's rule, and using equations (2)–(5) yields the following result:

$$\int_0^1 \phi_i \phi_i \, \mathrm{d}\xi = \left. \frac{[\Phi_1(\xi)]_0^1}{2(\omega^2 - k)(p^2 + 4\omega^2 - 4k)} \right|_{\omega = \omega_i}, \tag{15}$$

where

$$\begin{aligned} \Phi_1(\xi) = {} & 2(\omega^2 - k)(\phi(3\phi''' + (\omega^2 - k)\xi\phi) + \phi''(\xi\phi'' - \phi') - 2\xi\phi'\phi''') \\ & + p(\omega^2 - k)(\xi\phi(p\phi - 2\phi'') + 3(\xi\phi' - \phi)\phi') \\ & + p(\phi''' - p\phi')(\xi(\phi''' - p\phi') - \phi'' + p\phi). \end{aligned} \tag{16}$$

This result holds for all the mode functions satisfying equation (13) regardless of the sign of $p^2 + 4\omega^2 - 4k$. Letting $p = 0$ and $k = 0$ in equations (15) and (16) leads to the same result as given in references [2, 3].

As a second example, we will apply the present method to a uniform Timoshenko beam of finite length l. When considering a beam vibrating freely, we have the equations of motion as follows:

$$\rho A \frac{\partial^2 w}{\partial t^2} - \kappa G A \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} - \psi\right) = 0, \tag{17}$$

$$\rho A \frac{\partial^2 w}{\partial t^2} - EI \frac{\partial^2 \psi}{\partial x^2} - \kappa G A \left(\frac{\partial w}{\partial x} - \psi\right) = 0. \tag{18}$$

where G is the shear modulus, κ is the shear coefficient, ψ is the rotatory angle of the beam due to the bending moment, and the other symbols are the same as in the first example. The associated boundary conditions are assumed to be at $x = 0$,

$$EI \frac{\partial \psi}{\partial x} - K_{\theta 0}\psi = 0, \quad \kappa G A\left(\frac{\partial w}{\partial x} - \psi\right) - K_{T0} w = 0, \tag{19}$$

and at $x = l$,

$$EI \frac{\partial \psi}{\partial x} + K_{\theta 1}\psi = 0, \quad \kappa G A\left(\frac{\partial w}{\partial x} - \psi\right) + K_{T1} w = 0. \tag{20}$$

In addition to equation (9), we here introduce new dimensionless quantities such as $\varepsilon = E/\kappa G$ and $\eta = r/l$ where r is the radius of gyration of the cross-section. Then, equations

(17)–(20) become

$$\varepsilon\eta^2 \frac{\partial^2 w}{\partial \tau^2} - \left(\frac{\partial^2 w}{\partial \xi^2} - \frac{\partial (l\psi)}{\partial \xi}\right) = 0, \quad \varepsilon\eta^2 \left(\eta^2 \frac{\partial^2 (l\psi)}{\partial \tau^2} - \frac{\partial^2 (l\psi)}{\partial \xi^2}\right) - \left(\frac{\partial w}{\partial \xi} - (l\psi)\right) = 0, \tag{21}$$

$$(l\psi)' - k_{\theta 0}(l\psi) = 0, \quad w' - (l\psi) - \varepsilon\eta^2 k_{T0} w = 0 \tag{22}$$

and

$$(l\psi)' + k_{\theta 1}(l\psi) = 0, \quad w' - (l\psi) + \varepsilon\eta^2 k_{T1} w = 0. \tag{23}$$

Substituting $w = \phi(\xi)\mathrm{e}^{\mathrm{i}\omega\tau}$ and $(l\psi) = \varphi(\xi)\mathrm{e}^{\mathrm{i}\omega\tau}$ into two equations in equation (21), and combining the results yields the decoupled equations in the same form as equation (1),

$$\phi'''' + f\phi'' + g\phi = 0 \tag{24}$$

and

$$\varphi'''' + f\varphi'' + g\varphi = 0, \tag{25}$$

where $f = \eta^2(1+\varepsilon)\omega^2$ and $g = \omega^2(\varepsilon\eta^4\omega^2 - 1)$. Here, it should be noticed that the rotatory angle φ is related to the transverse deflection ϕ by the following equation:

$$\varphi = \phi' - \frac{\varepsilon\eta^2\omega^2}{g}(\phi''' + f\phi'). \tag{26}$$

Denoting the solutions of equations (24) and (25) with $\omega = \omega_i$ by ϕ_i and φ_i respectively, we have the relation as follows:

$$\int_0^1 (\phi_i\phi_j + \eta^2\varphi_i\varphi_j)\,\mathrm{d}\xi = \frac{[\phi_i(\varphi_j - \phi_j') - (\varphi_i - \phi_i')\phi_j - \varepsilon\eta^2(\varphi_i\varphi_j' - \varphi_i'\varphi_j)]_0^1}{\varepsilon\eta^2(\omega_j^2 - \omega_i^2)}. \tag{27}$$

The right-hand side of the above equation will be definite when either ε or η approaches zero because of equation (26). If the functions ϕ_i, φ_i, ϕ_j and φ_j satisfy the associated boundary conditions, the right-hand side of the above equation vanishes on condition that ω_i and ω_j are distinct, thus confirming orthogonality of the mode functions for a uniform Timoshenko beam. The differentiations of φ and φ' with respect to the parameter ω are necessary to evaluate the right-hand side of equation (27) for $\omega_i = \omega_j$. With the help of equation (26), they can be expressed as

$$\varphi_\omega = \phi_\omega' - \frac{\varepsilon\eta^2\omega^2}{g^2}\{g(\phi_\omega''' + f_\omega\phi' + f\phi_\omega') - 2\varepsilon\eta^4\omega^3(\phi''' + f\phi')\} \tag{28}$$

and

$$\varphi_\omega' = \phi_\omega'' + \frac{\varepsilon\eta^2\omega^2}{g}(g\phi_\omega + g_\omega\phi - 2\varepsilon\eta^4\omega^3\phi). \tag{29}$$

Using L'Hospital's rule in the limit when ω_j approaches ω_i, we obtain the following result:

$$\int_0^1 (\phi_i\phi_i + \eta^2\varphi_i\varphi_i)\,\mathrm{d}\xi = \left.\frac{[\Phi_2(\xi)]_0^1}{2g^2\omega}\right|_{\omega=\omega_i} \tag{30}$$

with

$$\Phi_2(\xi) = \omega^2(g(\phi''' + f\phi')\phi_\omega + q\phi) + (\phi'' + \varepsilon\eta^2\omega^2\phi)(g^2\phi'_\omega + \varepsilon\eta^2\omega^2 q)$$
$$+ (\varepsilon\eta^2\omega^2(g\phi_\omega + g_\omega\phi - 2\varepsilon\eta^4\omega^3\phi) + g\phi''_\omega)(\varepsilon\eta^2\omega^2(\phi''' + f\phi') - g\phi'), \quad (31)$$

where

$$q = 2\varepsilon\eta^4\omega^3(\phi''' + f\phi') - g(\phi'''_\omega + f_\omega\phi' + f\phi'_\omega). \quad (32)$$

Substituting equations (2)–(5) into equations (31) and (32), we can express the right-hand side of equation (30) in terms of the boundary values of the function ϕ_i and its derivatives of higher order with respect to ξ. When both ε and η in equation (30) become zero, the resulting expression is equivalent to equation (15) with the conditions of $p = 0$ and $k = 0$. Although equation (30) seems to be still complicated, the computing procedure for the normalization integral of the mode functions of a uniform Timoshenko beam will be remarkably simplified because (i) the present result does not include the function φ but includes only the function ϕ; (ii) the present result holds for all of the functions ϕ_i that will change their expressions in accordance with the sign of $\varepsilon\eta^4\omega_i^2 - 1$, and (iii) it is quite simple to incorporate the present result in a computer program. When we let $\varepsilon = 0$ in the present rule, equation (30) becomes the formula for the normalizing factor of a uniform Rayleigh beam which includes only the effect of the rotatory inertia.

Furthermore, we can easily rewrite the present results in terms of the variables in the transfer matrix [8] using the following relations:

$$\phi' = \varphi + \varepsilon\eta^2\bar{Q},$$
$$\phi'' = -\varepsilon\eta^2\omega^2\phi - \bar{M}$$

and

$$\phi''' = -(1 + \varepsilon)\eta^2\omega^2\varphi - (1 + \varepsilon\eta^4\omega^2)\bar{Q},$$

with $\bar{M} = Ml^2/EI$ and $\bar{Q} = Ql^3/EI$ where M and Q are the bending moment and the shear force, respectively. Therefore, the present results are also available in dynamic analysis using the transfer matrix method.

4. CONCLUSION

A useful method for evaluating the normalization integral of the mode functions of a vibrating uniform beam has been presented. Hence, one can avoid either lengthy analytical integration or complicated numerical integration. The present method is based on only the property of the fundamental governing differential equation and it can be applied to a beam whose deflection in a steady-state vibration satisfies equation (1) including a uniform Timoshenko beam theory. The present results will be available in various vibration problems such as stepped beams with intermediate elastic supports and multi-span continuous beams.

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